Jet Impinging on a Cart

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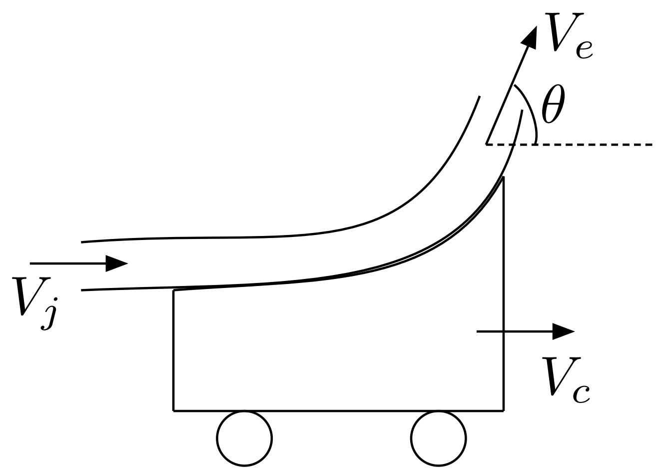
September 12, 2016



1 Case 1: Cart fixed

We will select a control volume that encloses and follows the water jet. We assume that the flow is steady, inviscid, and incompressible.

Mass balance:

$$\frac{\partial}{\partial t}\int_{\forall} \rho d\forall + \int_S \rho \vec{W}\cdot d\vec{A} = 0$$
$$\int_S \vec{W}\cdot d\vec{A} = 0$$
$$V_j A_j = V_e A_e$$

Assuming small elevation change (compared to contribution from velocity) and incompressible flow, from Bernoulli we have:

$$P_j + \frac{1}{2}\rho V_j^2 = P_e + \frac{1}{2}\rho V_e^2$$

This is a free jet so we know that $P_j = P_e$. Thus, from mass conservation we find: $V_j = V_e$.

Momentum balance:

$$\frac{\partial}{\partial t}\int_{\forall} \rho \vec{V} d\forall + \int_S \rho \vec{V}(\vec{W}\cdot d\vec{A}) = -\int_S p d\vec{A} + \int_S \overleftrightarrow{\tau}\cdot d\vec{A} + \sum \vec{F}_{other}$$

With a free jet the only force is that from the cart onto the fluid. Momentum in x-direction is

$$\begin{aligned}
\int_S \rho V_x(\vec{W}\cdot d\vec{A}) &= -R_x \\
\rho V_j(-V_jA_j) + \rho V_j\cos\theta(V_jA_j) &= -R_x \\
\Rightarrow R_x &= \rho V_j^2 A_j(1-\cos\theta)
\end{aligned}$$

2 Case 2: Cart moving at a constant speed

Mass balance is based on relative speed (recall $\vec{V} = \vec{V_c} + \vec{W}$). Flow is still steady. Thus, the mass balance does not change in terms of relative velocity.

$$|W_1| = |W_2|$$

Velocity vectors

$$\begin{aligned}
\vec{V_1} &= V_j\hat{x} \\
\vec{W_1} &= (V_j - V_c)\hat{x} \\
\vec{W_2} &= (V_j - V_c)\cos\theta\hat{x} + (V_j - V_c)\sin\theta\hat{y} \\
\vec{V_2} &= [(V_j - V_c)\cos\theta + V_c]\hat{x} + (V_j - V_c)\sin\theta\hat{y}
\end{aligned}$$

Momentum balance in x-direction:

$$\begin{aligned}
\int_S \rho V_x(\vec{W}\cdot d\vec{A}) &= -R_x \\
\rho V_j(-(V_j - V_c)A_j) + \rho[(V_j - V_c)\cos\theta + V_c]((V_j - V_c)A_j) &= -R_x \\
\Rightarrow R_x &= \rho(V_j - V_c)^2 A_j(1-\cos\theta)
\end{aligned}$$

Note: we could have reached this conclusion much more easily by solving in an inertial reference frame that moves with the cart, rather than using a stationary control volume. In that case, we just replace V_j with $V_j - V_c$.

3 Case 3: Cart accelerating

Everything is the same except for the unsteady terms. We usually assume that the mass of the cart is much larger than the mass of the water and so ignore any unsteady terms for the water. The mass of the cart does not change so the mass balance stays the same. The momentum of the cart does change. We have the additional term:

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{V\!\!\!\!-} \rho\vec{V}\,dV\!\!\!\!- \\
&\frac{\partial}{\partial t}(M_cV_c) \\
&M_c\frac{dV_c}{dt}
\end{aligned}$$

Adding this terms yields:

$$R_x = M_c\frac{dV_c}{dt} + \rho(V_j - V_c)^2 A_j(1-\cos\theta)$$

If there is no reaction force, we have an ODE we can solve for the velocity of the cart:

$$\frac{dV_c}{dt} = \frac{\rho(V_j - V_c)^2 A_j(\cos\theta - 1)}{M_c}$$

In this case, we cannot use a control volume moving with the cart because it is non-inertial (accelerating).