Matching Mach and Reynolds Number

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In Star-CCM+ the freestream boundary condition can be specified with Mach number, pressure, and temperature. The important nondimensional parameters we need to match are Mach number and Reynolds number. Setting the Mach number is obvious, but what pressure and temperature do we use in order to match Reynolds number?

Let's start with the definitions of Mach number and Reynolds number (based on chord in this case):

$$M_\infty = \frac{V_\infty}{a}, \quad Re = \frac{\rho V_\infty c}{\mu}$$

How does temperature and pressure affect these equations? The speed of sound and the dynamic viscosity are both functions of temperature, and the density is a function of pressure and temperature through a thermodynamic equation of state (generally the ideal gas law). In other words:

$$M_\infty = \frac{V_\infty}{a(T)}, \quad Re = \frac{\rho(P,T) V_\infty c}{\mu(T)}$$

The freestream velocity appears in both equations, so solve for it in one equation and plug it into the other.

$$Re = \frac{\rho(P,T) M_\infty a(T) c}{\mu(T)}$$

We note that we have a degree of freedom. We can either choose a pressure and then solve for the corresponding temperature that satisfies the equation, or we can specify temperature and then choose the corresponding pressure that satisfies the equation. Intuitively that should make sense. We should be able to match Mach and Reynolds Number at any condition by appropriately choosing the other variables.

The density has a simple relationship between pressure and temperature through the ideal gas law:

$$P = \rho R T$$

where the specific gas constant $R = 286.9$ J/(kg-K) for air. Substituting that into our main equation:

$$Re = \frac{P M_\infty a(T) c}{R T \mu(T)}$$

The speed of sound also has a simple relationship with temperature:

$$a = \sqrt{\gamma R T}$$

where $\gamma = 1.4$ for an ideal diatomic gases (and air is essentially entirely composed of diatomic gases). Substituting in:

$$Re = \frac{P M_\infty \sqrt{\gamma R T} c}{R T \mu(T)} = \frac{P M_\infty \sqrt{\gamma} c}{\sqrt{RT} \mu(T)}$$

The only thing we haven't substituted in is the dynamic viscosity dependence on temperature. We won't directly substitute in just because it is a little longer. For an ideal gas, the dynamic viscosity can be found from Sutherland's law:

$$\mu = \mu_{ref} \left(\frac{T}{T_{ref}}\right)^{3/2} \frac{T_{ref} + S}{T + S}$$

where $T_{ref} = 273.15$, $S = 110.4$, $\mu_{ref} = 1.716 \times 10^{-5}$ kg/(m-s).

We can see that the easiest way to solve this equation is to choose T, and then compute P (note that the units for T are Kelvin in all of these equations):

$$P = \frac{Re\ \mu(T)\sqrt{RT}}{M_\infty c \sqrt{\gamma}}$$

By choosing a T, we know everything on the right hand side and can directly solve for P.

The opposite approach is also possible, but is more work.

$$\sqrt{T}\mu(T) = \frac{PM_\infty c\sqrt{\gamma}}{Re\sqrt{R}}$$

We actually can solve for T explicitly through a quadratic function, but it's messy and is easier just to solve the above numerically as a root finding problem:

$$f(T) = \sqrt{T}\mu(T) - \frac{PM_\infty c\sqrt{\gamma}}{Re\sqrt{R}} = 0$$

We can use fzero in Matlab to find the T that satisfies $f(T) = 0$ (where again T is in Kelvin).

Either approach is fine, but remember that when you set the temperature and pressure in your boundary condition, you should also use them to set the initial conditions (or at least something close). If your initial conditions are very far from the steady state solution, you may have numerical issues and a difficult time converging. If you change pressure, it may be easiest to just change the reference pressure and then your gauge pressure can remain at 0 elsewhere.